Eddy Damped Quasi-Normal Markovian (EDQNM) Closure Model for Turbulence

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Turbulence is ubiquitous in the universe, yet we don't have a good model to understand it. Closure theory is one of the good tools to study turbulence, and Eddy Damped Quasi-Normal Markovian (EDQNM) model is the most popular closure model among various closure models nowadays, because EDQNM avoids many unphysical features and fits the experiment very well, especially the Markovianization assumption guarantees the realizability.

INTRODUCTION

Turbulence is one of the most important remaining mystery in classical mechanics. The story starts from the (incompressible) Navier-Stokes Equation, which is the governing equation for fluids:

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{u} \tag{1}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is the velocity field, p is pressure, ρ is density, and ν is the viscosity. It is well known that mathematicians are facing a huge challenge solving Navier-Stokes Equation.

When the dimensionless parameter Reynolds number, which is the ratio of momentum forces to viscous force, is larger than some specific threshold, the flow will inevitably grow into turbulence. Turbulence is ubiquitous in the universe, yet we don't fully understand it.

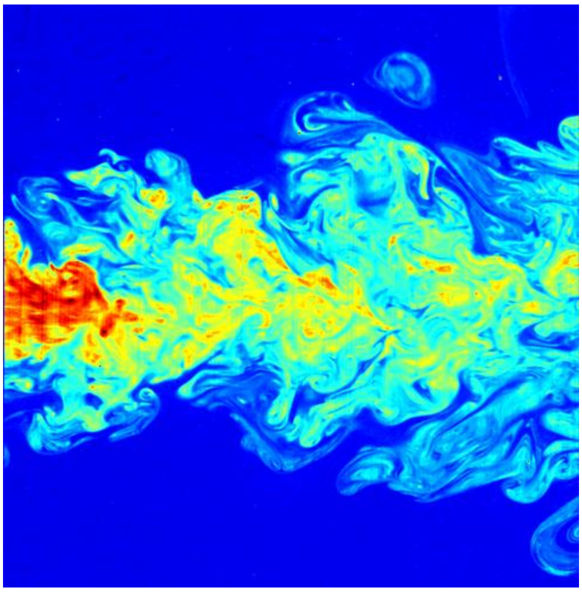

FIG. 1: A demonstration of turbulence.[1]

Direct Numerical Simulation (DNS) can be used to study turbulence, but it requires a huge amount of computing power to simulate an even small system, and practically impossible to simulate a real life problem.

A successful phenomenological theory for turbulence is the K41 Theory. Kolmogorov studied the energy spectrum of the homogeneous isotropic turbulence in 1941, he found that dissipation only occurs in small scale, and there is an inertial range in k space for turbulence (see Fig. 2), in which range the energy is neither injected nor dissipated, energy is only transferred from large scale to small scale. He obtained the famous $-5/3$ exponent of the turbulence energy spectrum in its inertial range:

$$E_k = C\varepsilon^{2/3}k^{-5/3} \tag{2}$$

where E_k is the energy spectrum, ε is a constant energy dissipation rate, k is the wave number, and C is a universal constant which value can not be derived from K41 theory itself.

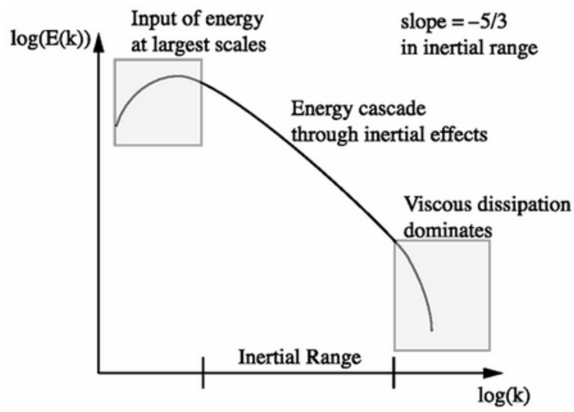

FIG. 2: K41 Theory: inertial range.[2]

K41 Theory is a big success, because it fits the experiments very well and the results are universal and robust. However there are two major concerns: 1. it doesn't consider a phenomenon called intermittency at all, since this is not related to the topic, it won't be discussed here; 2. K41 theory is too simple, it is a phenomenological theory about energy spectrum of turbulence, mainly derived by dimensional analysis, and is only able to give us limited information, it does not give us insight into the dynamics or transport properties of turbulences. Besides, some people believe we should be able to develop a theory to calculate the universal constant C in K41 Theory from first principle without any free parameter.

So more theoretical studies about the dynamics and transport of turbulences are done beyond the K41 Theory. The two most successful methods are Renormalization Group (RNG) Theory and Closure Theory. In this Letter, I will focus on the latter. I will first do a brief introduction to Closure Theory, then overview two unsuccessful closure models Quasi-Normal Approximation (QNA) and Direct Interaction Approximation (DIA), and finally review the currently most accepted closure model Eddy Damped Quasi-Normal Markovian (EDQNM) model.

CLOSURE THEORY

To demonstrate what is the closure theory, we rewrite Navier-Stokes Equation to a more general form that many nonlinear systems have:

$$\frac{du_i}{dt} + \nu k^2 u_i = \sum_{mn} M_{imn} u_m u_n \tag{3}$$

where the coupling coefficient M_{imn} are functions of wave vectors k,p,q forming a triangle $k + p + q = 0$. By multiplying u_j and averaging the equation (and doing symmetrization), we get:

$$\frac{d}{dt}\langle u_i u_j\rangle + \nu k^2 \langle u_i u_j\rangle = \sum_{mn}(M_{imn}\langle u_j u_m u_n\rangle + M_{jmn}\langle u_i u_m u_n\rangle) \tag{4}$$

Similarly we can get an equation for the triple-correlation:

$$\frac{d}{dt}\langle u_i u_j u_k\rangle + \nu k^2 \langle u_i u_j u_k\rangle = \sum_{mn}(M_{imn}\langle u_j u_k u_m u_n\rangle + M_{jmn}\langle u_i u_k u_m u_n\rangle + M_{kmn}\langle u_i u_j u_m u_n\rangle) \tag{5}$$

It's easy to see if we continue this process, whenever we want an equation for the n-th order correlation function, we have a n+1-th order correlation function in the r.h.s.. Thus we get an infinite hierarchy of equations, replacing the original one.

In order to get some knowledge of the system, we can't solve the infinite set of equations, just as we are not able to solve the original Navier-Stokes Equation. We must truncate this set of equations by a model, each reasonable model is called a closure model.

For example, in thermal dynamics, people usually assume Maxwellian velocity distribution (or say normal distribution) for molecular motions in order to close a hierarchy of equations.

If we assume the probability density function (pdf) of u_i to be Gaussian or Normal, then

$$\langle u_i u_j u_k\rangle = 0 \tag{6}$$

$$\langle u_i u_j u_k u_l\rangle = \langle u_i u_j\rangle\langle u_k u_l\rangle + \langle u_i u_k\rangle\langle u_j u_l\rangle + \langle u_i u_l\rangle\langle u_k u_j\rangle \tag{7}$$

Now that we are able to use a lower order cumulant to express a higher order cumulant, the hierarchy of equations are closed.

Sadly, this simple normal distribution model is not suitable for turbulence, because otherwise the r.h.s. of equation (4) will be 0, which means all nonlinear effects are gone. So we need other closure models.

QUASI-NORMAL APPROXIMATION (QNA)

A natural improvement of the simple normal distribution model is to assume "quasi-normal" property for the pdf: we only assume the equation for 4th order cumulant (7) is true, and discard the equation for 3rd order cumulant (6), instead we calculate it by (7) and (5). This approach is called Quasi-Normal Approximation (QNA).

After some tedious but straightforward calculation, we can get the equation for energy spectrum $E_k = \langle|u_k|^2\rangle$:

$$\partial_t E_k + 2\nu k^2 E_k = \int_0^t \mathrm{d}t' \int_{\Delta_k} \mathrm{d}p\,\mathrm{d}q\, G_{kpq}(t,t')\frac{a_{kpq}}{q}[k^2 E_p(t')E_q(t') - p^2 E_k(t')E_q(t')] \tag{8}$$

where k,p,q are wave numbers, Δ_k is the k range that k,p,q can form a triangle, $a_{kpq} = xy + z^3$ where x,y,z are cosines of the angles in the triangle facing the sides k,p,q respectively, and $G_{kpq}(t,t')$ is the Green's function of l.h.s. of (5).

In the QNA model, the Green's function is

$$G_{kpq}(t,t') = e^{-(\nu k^2+\nu p^2+\nu q^2)(t-t')} \tag{9}$$

The r.h.s. of (8) contains two parts: a coherent part proportional to E_k representing the turbulent viscosity, and a incoherent part acting as a noisy source or sink of turbulent energy.

The major problem with QNA model is *realizability*: basic probabilistic inequalities may be violated. Some simulations verified this: in QNA model the energy spectrum might be negative, which is unphysical.

DIRECT INTERACTION APPROXIMATION (DIA)

Kraichnan noticed the realizability problem of the naive QNA, and he proposed a closure model called Direct Interaction Approximation (DIA).

The failure of QNA model is mainly due to the excessively long memory time $\sim (\nu k^2)^{-1}$ in (9), which is much longer than the corresponding eddy turnover time. So the memory time must be replaced by some turbulent eddy damping time $\tilde{\mu}$ instead of the original viscous damping $\mu = \nu k^2$. This is a main spirit of DIA model.

This field theory based model is quite complicated, so I won't go into details about it. DIA is not only able solve the realizability problem of QNA, it is also able to self-consistently calculate the responsible function, and thus calculate the universal constant C without any free parameter.

Unfortunately, there is a major failure in DIA model: it is not consistent with K41 Theory. Instead of (2), the energy spectrum derived from DIA is

$$E_k = C'(\varepsilon v_0)^{1/2} k^{-3/2} \tag{10}$$

Where C' is a universal constant like the one in K41 Theory, and v_0 is the r.m.s. turbulent velocity. The exponent in K41 Theory and experiments is $-5/3$, not $-3/2$.

Another serious defect of DIA is that DIA is not invariant under *random Galilean transformations*.

EDDY DAMPED QUASI-NORMAL MARKOVIAN (EDQNM) MODEL

Eddy Damped Quasi-Normal Markovian (EDQNM) Model is a DIA based closure model. Instead of a self-consistent derivation of the response function, EDQNM introduce a phenomenological eddy damping rate:

$$\tilde{\mu} \sim \left(\int_0^k p^2 E_p \, \mathrm{d}p\right)^{1/2} \tag{11}$$

and then the Green function in (9) becomes:

$$\tilde{G}_{kpq}(t,t') = e^{-\int_{t'}^{t} \mathrm{d}\tau \, [\tilde{\mu}_k(\tau)+\tilde{\mu}_p(\tau)+\tilde{\mu}_q(\tau)]} \tag{12}$$

There's an arbitrariness in (11), because we can choose the way to renormalize it into (12), and the parameter can be adjusted to fit the K41 constant C. So this approach actually sacrificed the systematic theory free of adjustable parameter.

Since now the effective time domain in the integral is the eddy interaction time $\tilde{\mu}_k^{-1}$, which is the shortest time scale, changing $E_k(t')$ in (8) to $E_k(t)$ wouldn't change the evolution of E_k significantly. This process is called *Markovianization*, which means ignoring any memory effect in the system. It can be proven that, Markovianization is the sufficient (but not necessary) condition to make the equations realizable.

After doing the Markovianization, we can integrate the Green's function first, and the closure equation can be simplified:

$$\partial_t E_k + 2\nu k^2 E_k = \int_{\Delta_k} \mathrm{d}p \, \mathrm{d}q \, \theta_{kpq} \frac{a_{kpq}}{q} [k^2 E_p(t) E_q(t) - p^2 E_k(t) E_q(t)] \tag{13}$$

where $\theta_{kpq} = \int_0^t \mathrm{d}t' \, \tilde{G}_{kpq}(t,t') = [1-\exp(-\tilde{\mu}_{kpq})]/\tilde{\mu}_{kpq}$ is the triad relaxation time, $\tilde{\mu}_{kpq} = \tilde{\mu}_k + \tilde{\mu}_p + \tilde{\mu}_q$.

(13) is the equation for EDQNM. This equation can also be expressed in the form of a Langevin equation:

$$\frac{d}{dt}\hat{u}_k = -\alpha_k \hat{u}_k + \hat{q}_k \tag{14}$$

where $\hat{u}_k$ is a stochastic variable, which in general is not identical to the velocity mode u_k, but they have the same variance: $\langle \hat{u}_k^2 \rangle = \langle u_k^2 \rangle$; $\hat{q}_k$ is white noise stochastic variable; and α_k is a non-stochastic damping rate.

EDQNM has the ability to solve both the realizability problem and the random Galilean transformation problem, and it is also consistent with K41 Theory.

After EDQNM closure model was developed, more models are made to keep all its good features while eliminate its free parameter, such as Test Field Model (TFM) and Lagrangian History Direct Interaction Approximation (LHDIA). However, they are too sophisticated to be useful for most turbulence study. So the most widely accepted closure model is still EDQNM.

CONCLUSION AND DISCUSSION

Among various closure models for turbulence, EDQNM is simple, realizable, invariant under random Galilean transformation, and consistent with K41 Theory, which makes it a very good model to study the dynamics and transport properties of turbulences.

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